Algorithms ROBERT SEDGEWICK | KEVIN WAYNE

http://algs4.cs.princeton.edu

3.2 BINARY SEARCH TREES

- BSTs
- ordered operations
- deletion

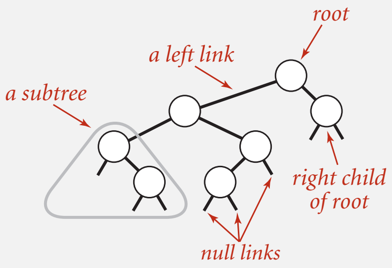
http://algs4.cs.princeton.edu

3.2 BINARY SEARCH TREES

- BSTs
- ordered operations
- deletion

Binary search trees

Definition. A BST is a binary tree in symmetric order.

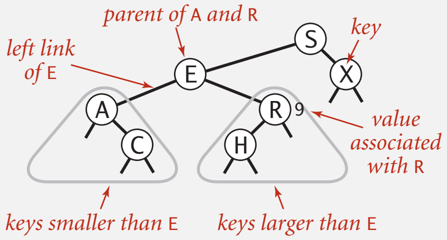


A binary tree is either:

- Empty.
- Two disjoint binary trees (left and right).

Symmetric order. Each node has a key, and every node's key is:

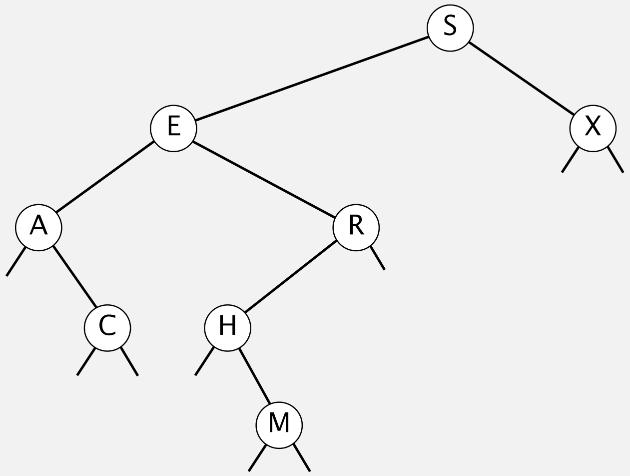
- Larger than all keys in its left subtree.
- Smaller than all keys in its right subtree.



Binary search tree demo

Search. If less, go left; if greater, go right; if equal, search hit.

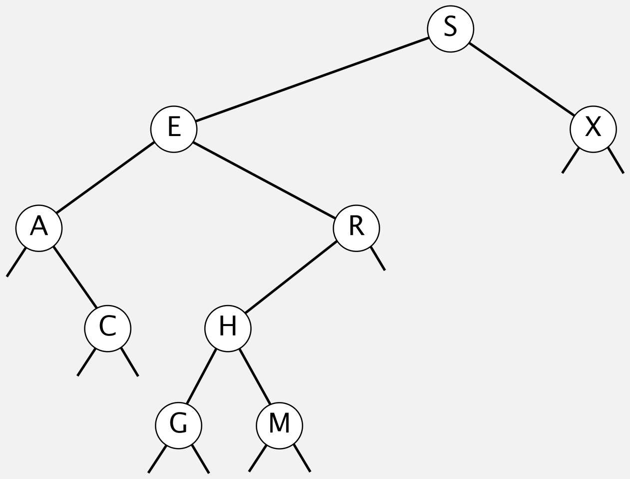
successful search for H



Binary search tree demo

Insert. If less, go left; if greater, go right; if null, insert.

insert G



BST representation in Java

Java definition. A BST is a reference to a root `Node`.

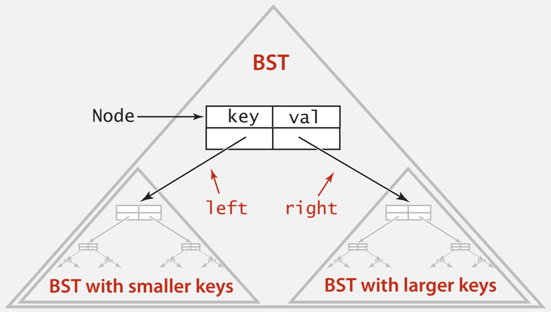
A `Node` is composed of four fields:

- A `Key` and a `Value`.
- A reference to the left and right subtree.

smaller keys   larger keys

```
private class Node
{
   private Key key;
   private Value val;
   private Node left, right;
   public Node(Key key, Value val)
   {
      this.key = key;
      this.val = val;
   }
}
```

Key and Value are generic types; Key is Comparable



Binary search tree

BST implementation (skeleton)

```
public class BST<Key extends Comparable<Key>, Value>
{
    private Node root;

   private class Node
   {  /* see previous slide */  }

   public void put(Key key, Value val)
   {  /* see next slides */  }

   public Value get(Key key)
   {  /* see next slides */  }

   public void delete(Key key)
   {  /* see next slides */  }

   public Iterable<Key> iterator()
   {  /* see next slides */  }

}
```

root of BST

BST search: Java implementation

Get. Return value corresponding to given key, or `null` if no such key.

```
public Value get(Key key)
{
   Node x = root;
   while (x != null)
   {
      int cmp = key.compareTo(x.key);
      if      (cmp  < 0) x = x.left;
      else if (cmp  > 0) x = x.right;
      else if (cmp == 0) return x.val;
   }
   return null;
}
```

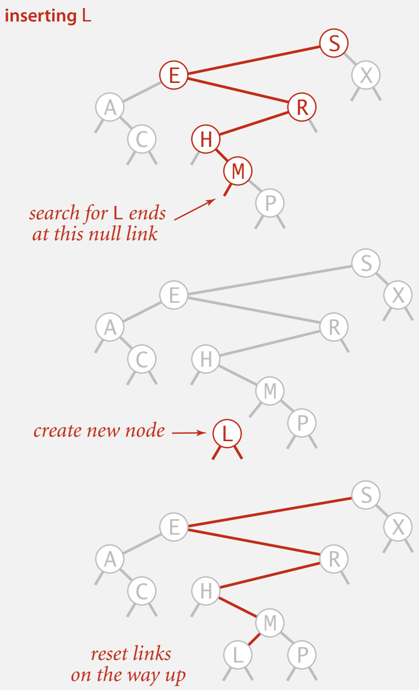
Cost. Number of compares is equal to 1 + depth of node.

BST insert

Put. Associate value with key.

Search for key, then two cases:

- Key in tree ⇒ reset value.
- Key not in tree ⇒ add new node.



Insertion into a BST

BST insert: Java implementation

Put. Associate value with key.

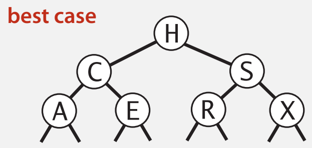
concise, but tricky, recursive code; read carefully!

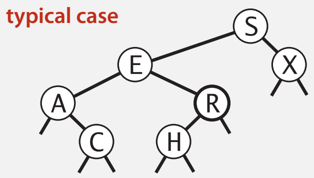
```
public void put(Key key, Value val)
{  root = put(root, key, val);  }

private Node put(Node x, Key key, Value val)
{
   if (x == null) return new Node(key, val);
   int cmp = key.compareTo(x.key);
   if      (cmp  < 0)
      x.left  = put(x.left,  key, val);
   else if (cmp  > 0)
      x.right = put(x.right, key, val);
   else if (cmp == 0)
      x.val = val;
   return x;
}
```

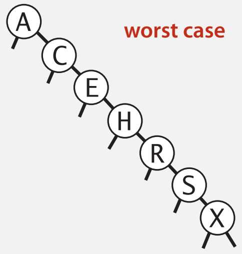
Cost. Number of compares is equal to 1 + depth of node.

Tree shape

- Many BSTs correspond to same set of keys.
- Number of compares for search/insert is equal to 1 + depth of node.



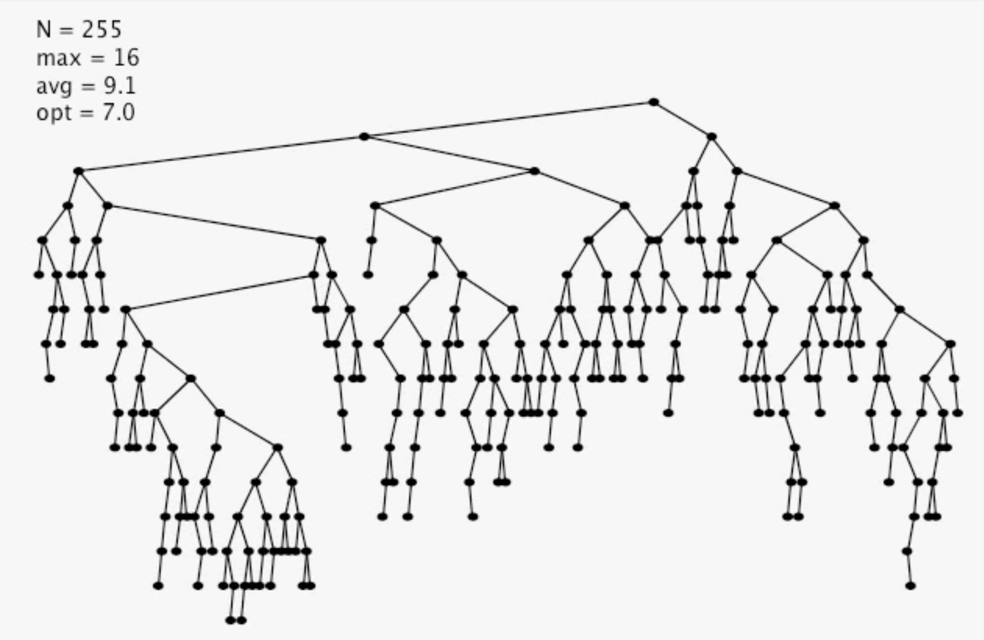




Bottom line. Tree shape depends on order of insertion.

BST insertion: random order visualization

Ex. Insert keys in random order.



Sorting with a binary heap

Q. What is this sorting algorithm?

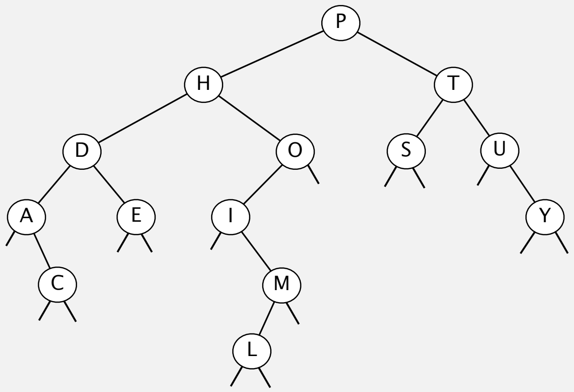
```
0. Shuffle the array of keys.
1. Insert all keys into a BST.
2. Do an inorder traversal of BST.
```

A. It's not a sorting algorithm (if there are duplicate keys)!

Q. OK, so what if there are no duplicate keys?
Q. What are its properties?

Correspondence between BSTs and quicksort partitioning

0	1	2	3	4	5	6	7	8	9	10	11	12	13
P	S	E	U	D	O	M	Y	T	H	I	C	A	L
P	S	E	U	D	O	M	Y	T	H	I	C	A	L
H	L	E	A	D	O	M	C	I	P	T	Y	U	S
D	C	E	A	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	I	M	L	O	P	T	Y	U	S
A	C	D	E	H	I	M	L	O	P	T	Y	U	S
A	C	D	E	H	I	L	M	O	P	T	Y	U	S
A	C	D	E	H	I	L	M	O	P	T	Y	U	S
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y

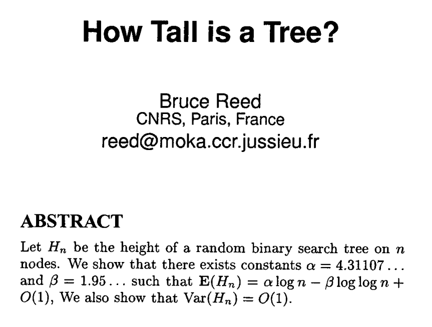


Remark. Correspondence is 1–1 if array has no duplicate keys.

BSTs: mathematical analysis

Proposition. If N distinct keys are inserted into a BST in random order, the expected number of compares for a search/insert is $\sim 2 \ln N$.
Pf. 1–1 correspondence with quicksort partitioning.

Proposition. [Reed, 2003] If N distinct keys are inserted in random order, expected height of tree is $\sim 4.311 \ln N$.



How Tall is a Tree?

Bruce Reed
CNRS, Paris, France
reed@moka.ccr.jussieu.fr

ABSTRACT

Let H_n be the height of a random binary search tree on n nodes. We show that there exists constants $\alpha = 4.31107\ldots$ and $\beta = 1.95\ldots$ such that $\mathbf{E}(H_n) = \alpha \log n - \beta \log\log n + O(1)$, We also show that $\mathrm{Var}(H_n) = O(1)$.

But… Worst-case height is N.
[exponentially small chance when keys are inserted in random order]

ST implementations: summary

implementation	guarantee		average case		operations on keys
	search	insert	search hit	insert	
sequential search (unordered list)	N	N	½ N	N	`equals()`
binary search (ordered array)	$\lg N$	N	$\lg N$	½ N	`compareTo()`
BST	N	N	$1.39 \lg N$	$1.39 \lg N$	`compareTo()`

Why not shuffle to ensure a (probabilistic) guarantee of 4.311 ln N?

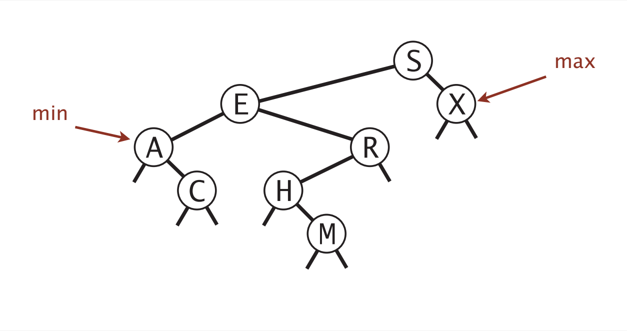
http://algs4.cs.princeton.edu

3.2 BINARY SEARCH TREES

- BSTs
- *ordered operations*
- deletion

Minimum and maximum

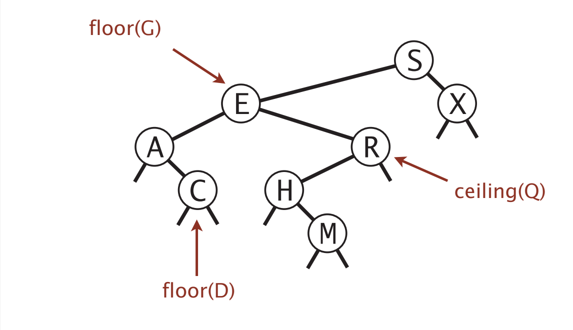
Minimum. Smallest key in table.
Maximum. Largest key in table.



Q. How to find the min / max?

Floor and ceiling

Floor. Largest key ≤ a given key.
Ceiling. Smallest key ≥ a given key.



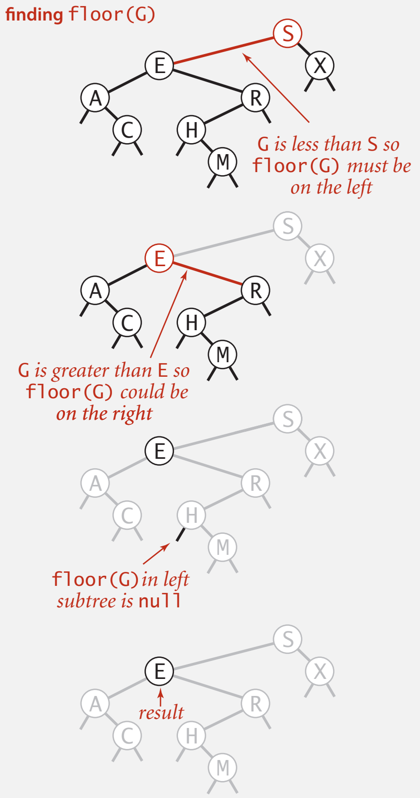
Q. How to find the floor / ceiling?

Computing the floor

Case 1. [*k* equals the key in the node]
The floor of *k* is *k*.

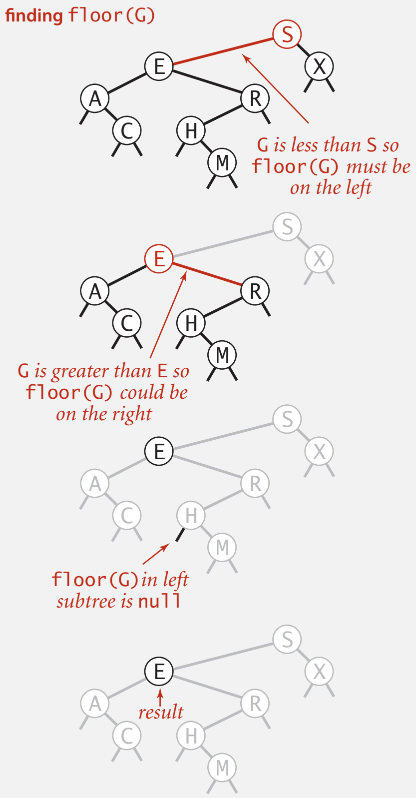
Case 2. [*k* is less than the key in the node]
The floor of *k* is in the left subtree.

Case 3. [*k* is greater than the key in the node]
The floor of *k* is in the right subtree
(if there is any key ≤ *k* in right subtree);
otherwise it is the key in the node.



Computing the floor

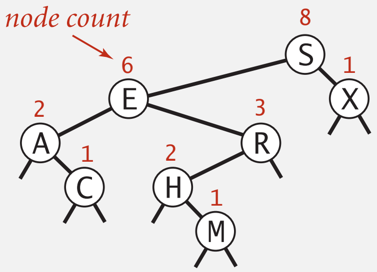
```
public Key floor(Key key)
{
   Node x = floor(root, key);
   if (x == null) return null;
   return x.key;
}
private Node floor(Node x, Key key)
{
   if (x == null) return null;
   int cmp = key.compareTo(x.key);

   if (cmp == 0) return x;

   if (cmp < 0)  return floor(x.left, key);

   Node t = floor(x.right, key);
   if (t != null) return t;
   else           return x;

}
```



Rank and select

Q. How to implement `rank()` and `select()` efficiently?

A. In each node, we store the number of nodes in the subtree rooted at that node; to implement `size()`, return the count at the root.



BST implementation: subtree counts

```
private class Node
{
   private Key key;
   private Value val;
   private Node left;
   private Node right;
   private int count;
}
```

number of nodes in subtree

```
public int size()
{  return size(root);  }

private int size(Node x)
{
   if (x == null) return 0;
   return x.count;
}
```

ok to call when x is null

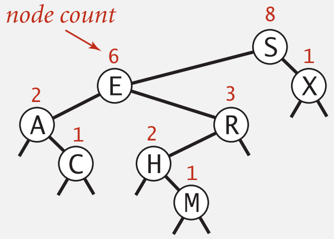
```
private Node put(Node x, Key key, Value val)
{
   if (x == null) return new Node(key, val, 1);
   int cmp = key.compareTo(x.key);
   if      (cmp  < 0) x.left  = put(x.left,  key, val);
   else if (cmp  > 0) x.right = put(x.right, key, val);
   else if (cmp == 0) x.val = val;
   x.count = 1 + size(x.left) + size(x.right);
   return x;
}
```

initialize subtree count to 1

Rank

Rank. How many keys < *k* ?

Easy recursive algorithm (3 cases!)

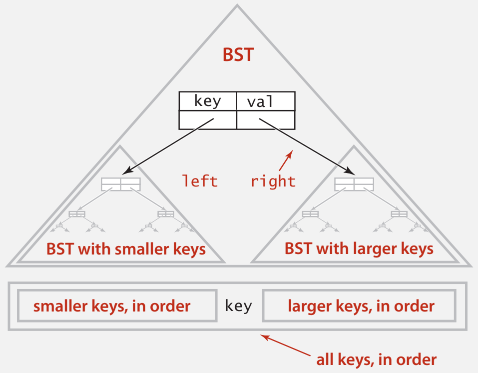


```
public int rank(Key key)
{  return rank(key, root);  }

private int rank(Key key, Node x)
{
   if (x == null) return 0;
   int cmp = key.compareTo(x.key);
   if      (cmp  < 0) return rank(key, x.left);
   else if (cmp  > 0) return 1 + size(x.left) + rank(key, x.right);
   else if (cmp == 0) return size(x.left);
}
```

Inorder traversal

- Traverse left subtree.
- Enqueue key.
- Traverse right subtree.

```
public Iterable<Key> keys()
{
    Queue<Key> q = new Queue<Key>();
    inorder(root, q);
    return q;
}

private void inorder(Node x, Queue<Key> q)
{
   if (x == null) return;
   inorder(x.left, q);
   q.enqueue(x.key);
   inorder(x.right, q);
}
```



Property. Inorder traversal of a BST yields keys in ascending order.

BST: ordered symbol table operations summary

	sequential search	binary search	BST
search	N	$\lg N$	h
insert	N	N	h
min / max	N	1	h
floor / ceiling	N	$\lg N$	h
rank	N	$\lg N$	h
select	N	1	h
ordered iteration	$N \log N$	N	N

h = height of BST
(proportional to log N
if keys inserted in random order)

order of growth of running time of ordered symbol table operations

---

http://algs4.cs.princeton.edu

3.2 BINARY SEARCH TREES

- *BSTs*
- *ordered operations*
- *deletion*

ST implementations: summary

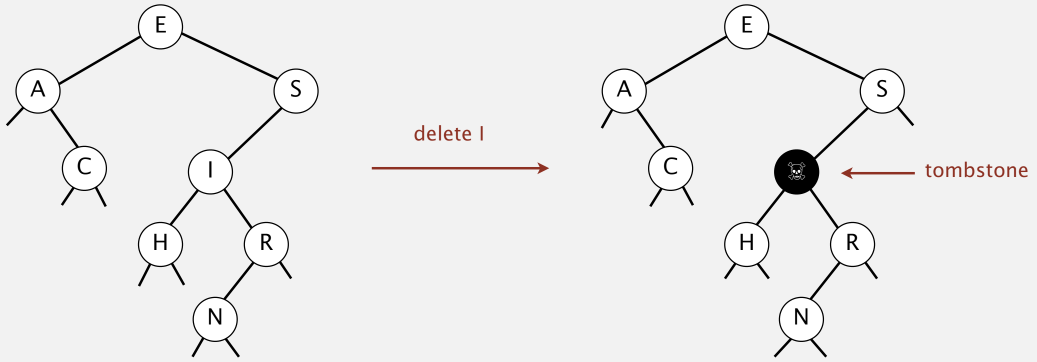
implementation	guarantee			average case			ordered ops?	operations on keys
	search	insert	delete	search hit	insert	delete		
sequential search (linked list)	N	N	N	½ N	N	½ N		`equals()`
binary search (ordered array)	$\lg N$	N	N	$\lg N$	½ N	½ N	✔	`compareTo()`
BST	N	N	N	$1.39 \lg N$	$1.39 \lg N$	? ? ?	✔	`compareTo()`

Next. Deletion in BSTs.

BST deletion: lazy approach

To remove a node with a given key:

- Set its value to `null`.
- Leave key in tree to guide search (but don't consider it equal in search).



Cost. $\sim 2 \ln N'$ per insert, search, and delete (if keys in random order), where N' is the number of key-value pairs ever inserted in the BST.

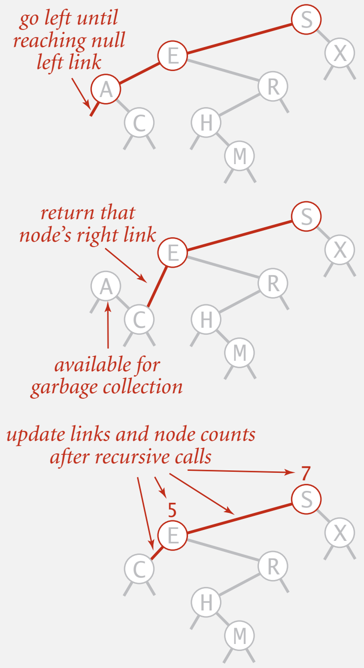
Unsatisfactory solution. Tombstone (memory) overload.

Deleting the minimum

To delete the minimum key:

- Go left until finding a node with a null left link.
- Replace that node by its right link.
- Update subtree counts.

```
public void deleteMin()
{  root = deleteMin(root);  }

private Node deleteMin(Node x)
{
   if (x.left == null) return x.right;
   x.left = deleteMin(x.left);
   x.count = 1 + size(x.left) + size(x.right);
   return x;
}
```



go left until reaching null left link

return that node's right link

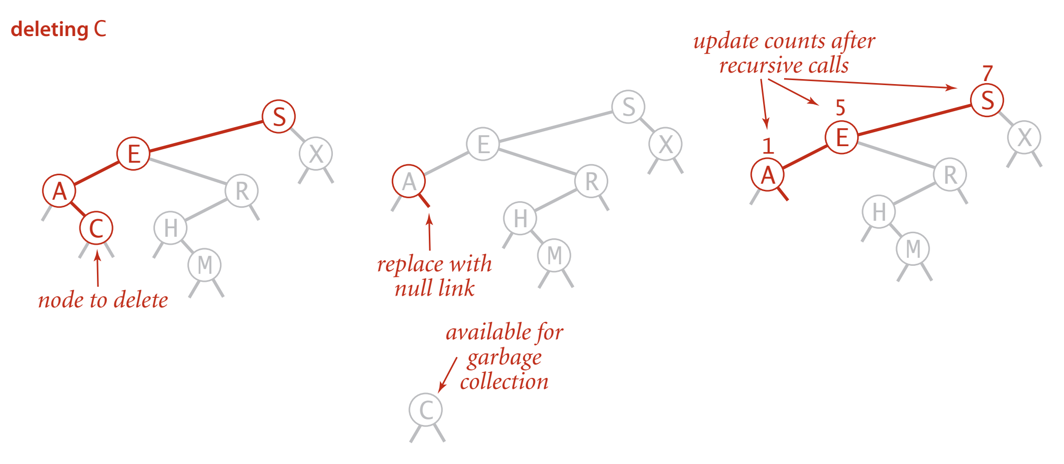
available for garbage collection

update links and node counts after recursive calls

Hibbard deletion

To delete a node with key `k`: search for node `t` containing key `k`.

Case 0. [0 children] Delete `t` by setting parent link to null.



node to delete

replace with null link

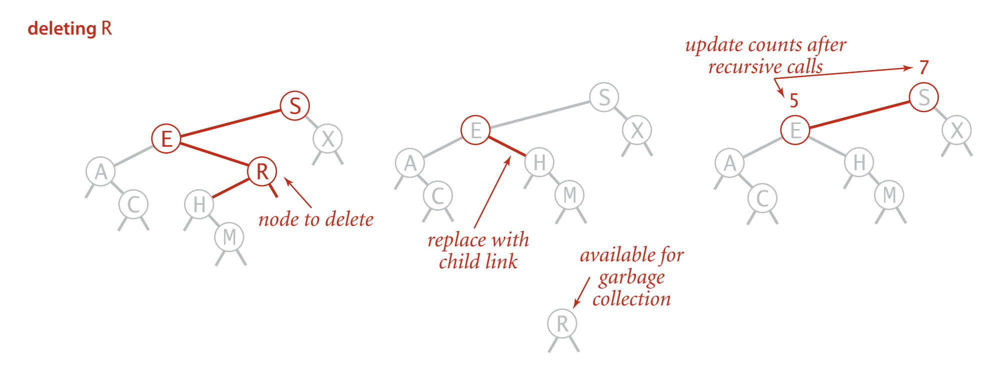
available for garbage collection

update counts after recursive calls

Hibbard deletion

To delete a node with key `k`: search for node `t` containing key `k`.

Case 1. [1 child] Delete `t` by replacing parent link.



node to delete

replace with child link

available for garbage collection

update counts after recursive calls

Hibbard deletion

To delete a node with key `k`: search for node `t` containing key `k`.

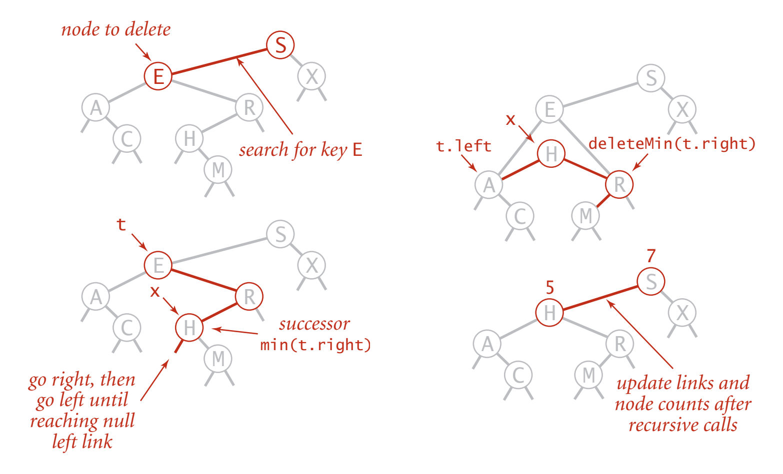
Case 2. [2 children]

- Find successor `x` of `t`.
- Delete the minimum in `t`'s right subtree.
- Put `x` in `t`'s spot.

x has no left child

but don't garbage collect x

still a BST



Hibbard deletion: Java implementation

```
public void delete(Key key)
{  root = delete(root, key);  }

private Node delete(Node x, Key key) {
   if (x == null) return null;
   int cmp = key.compareTo(x.key);
   if      (cmp < 0) x.left  = delete(x.left,  key);
   else if (cmp > 0) x.right = delete(x.right, key);
   else {
      if (x.right == null) return x.left;
      if (x.left  == null) return x.right;

      Node t = x;
      x = min(t.right);
      x.right = deleteMin(t.right);
      x.left = t.left;
   }
   x.count = size(x.left) + size(x.right) + 1;
   return x;
}
```

search for key

no right child

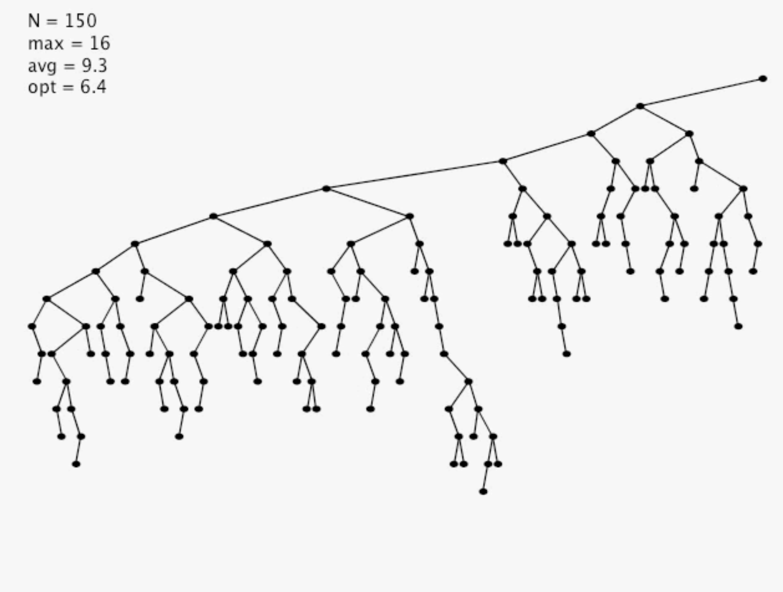
no left child

replace with successor

update subtree counts

Hibbard deletion: analysis

Unsatisfactory solution. Not symmetric.



Surprising consequence. Trees not random (!) ⇒ $\sqrt{N}$ per op.

Longstanding open problem. Simple and efficient delete for BSTs.

ST implementations: summary

implementation	guarantee			average case			ordered ops?	operations on keys
	search	insert	delete	search hit	insert	delete		
sequential search (linked list)	N	N	N	½ N	N	½ N		`equals()`
binary search (ordered array)	lg N	N	N	lg N	½ N	½ N	✔	`compareTo()`
BST	N	N	N	1.39 lg N	1.39 lg N	$\sqrt{N}$	✔	`compareTo()`

other operations also become √N if deletions allowed

Next lecture. Guarantee logarithmic performance for all operations.