ALGORITHM 245
TREESORT 3 [M1]
ROBERT W. FLOYD (Recd. 22 June 1964 and 17 Aug. 1964)

comment elements of the Boolean array a[1:n] may together be
considered as representing a logical vector value in the Gray
cyclic binary-code. [See e.g. Phister, M., Jr., Logical Design of
procedure changes one element of the array to form the next
code value in ascending sequence if the parity parameter s
= true or in descending sequence if s = false. The procedure
may also be applied to the classic “rings-o-seven” puzzle [see
K. E. Iverson, A Programming Language, p. 63, Ex. 1.5];

begin integer i,j; j := n + 1;
for i := n step -1 until 1 do if a[i] then begin s := -a s;
if s then a[i] := -a[i] else if j < n then a[j+1] := -a[j+1]
else a[n] := -a[n]
end graycode

ALGORITHM 246
GRAYCODE [Z]
J. BOOTHROYD* (Recd. 18 Nov. 1963)
English Electric-Leo Computers, Kidsgrove, Stoke-on-
Trent, England
* Now at University of Tasmania, Hobart, Tasmania, Aust.

procedure graycode (a) dimension: (n) parity: (s); value a,n,s;
Boolean array a; integer n; Boolean s;
comment elements of the Boolean array a[1:n] may together be
considered as representing a logical vector value in the Gray
cyclic binary-code. [See e.g. Phister, M., Jr., Logical Design of
procedure changes one element of the array to form the next
code value in ascending sequence if the parity parameter s
= true or in descending sequence if s = false. The procedure
may also be applied to the classic “rings-o-seven” puzzle [see
K. E. Iverson, A Programming Language, p. 63, Ex. 1.5];

begin integer i,j; j := n + 1;
for i := n step -1 until 1 do if a[i] then begin s := -a s;
if s then a[i] := -a[i] else if j < n then a[j+1] := -a[j+1]
else a[n] := -a[n]
end graycode

ALGORITHM 247
RADICAL-INVERSE QUASI-RANDOM POINT
SEQUENCE [G5]
J. H. HALTON AND G. B. SMITH (Recd. 24 Jan. 1964 and
21 July 1964)
Brookhaven National Laboratory, Upton, N. Y., and
University of Colorado, Boulder, Colo.

procedure QRPSH (K, N, P, Q, R, E); integer K, N; real array P, Q; integer array R; real E;
comment This procedure computes a sequence of N quasi-
random points lying in the K-dimensional unit hypercube
given by 0 < xi < 1, i = 1, 2, ... , K. The ith component of
the mth point is stored in Q[m,i]. The sequence is initiated by a
“zero-th point” stored in P, and each component sequence is
iteratively generated with parameter R[i]. E is a positive error-
parameter. K, N, E, and the P[i] and R[i] for i = 1, 2, ... , K,
are to be given.

The sequence is discussed by J. H. Halton in Num. Math. 2
(1960), 84-90. If any integer a is written in radix-R notation as
n = n_a ... n_1 n_0 . 0 = n_a R + n_1 R + ... + n_0 R^0,
and reflected in the radical point, we obtain the R-inverse func-
tion of n, lying between 0 and 1,
\[ \phi_E(n) = 0 . n_a n_1 n_2 ... n_m = n_a R^{m-1} + n_1 R^{m-2} \]
+ n_2 R^{m-3} + ... + n_m R^{m-m}.
The problem solved by this algorithm is that of giving a com-
 pact procedure for the addition of R^{-1}, in any radix R, to a frac-
tion, with downward “carry”.
If P[i] = \phi_{R[0]}(s), as will almost always be the case in practice,
with s a known integer, then Q[m,i] = \phi_{R[i]}(s+m). For quasi-
randomness (uniform limiting density), the integers R[i] must
be mutually prime.
For exact numbers, E would be infinitesimal positive. In prac-
tice, round-off errors would then cause the “carry” to be in-
correctly placed, in two circumstances. Suppose that the stored
number representing \phi_E(n) is actually \phi_E(n) + \Delta. (a) If |\Delta |
\geq E^{-1}, we see that the results of the algorithm become un-
predictable. It is necessary to stop before this event occurs. It may be delayed by working in multiple-length arithmetic. (b) If \( n = R^{\alpha + 1} - 1 \), so that \( \phi R(n) = 1 - R^{-1} \), and \( \alpha < 0 \), the computed successor of the stored value can be seen to be about \( R^{-\alpha} \), instead of \( R^{-\alpha-1} = \phi R(n+1) \). This error can be avoided, without disturbing the rest of the computation, by adopting a value of \( E \) greater than any \( |\Delta| \) which may occur, but smaller than the least \( (nE)^{-\alpha} \) (which is smaller than the least \( R^{-\alpha-1} \)) to be encountered.

Small errors in the \( P[i] \) will not affect the sequence. Any set of \( P[i] \) in the computer may be considered as a set of \( \phi R(n) \), for generally large and unequal integers \( i \), with small round-off errors. The arguments used in J. H. Halton's paper to establish the uniformity of the sequence of points

\[
[\phi R(n), \phi R(n), \ldots, \phi R(n)], \quad n = 1, 2, \ldots, N
\]
can be applied identically to the more general sequence

\[
[\phi R(s+n), \phi R(s+n), \ldots, \phi R(s+n)], \quad n = 1, 2, \ldots, N.
\]
Thus, theoretically, any "zero-th point" \( P \) will do. However, the difficulty described in (a) above limits us to the use of \( P[i] \) corresponding to relatively small integers \( i \); begin integer \( i, \text{m}; \) real \( r, f, g, h; \)

for \( i := 1 \) step 1 until \( k \) do

begin \( r := 1.0/R[i]; \)
for \( m := 1 \) step 1 until \( N \) do

\begin{center}
\textbf{begin if } m > 1 \textbf{ then } f := 1.0 - Q[m-1, i] \textbf{ else } f := 1.0 - P[i]; \textbf{ end if } \end{center}

\begin{center}
g := 1.0; \quad h := r; \quad \textbf{repeat: } \end{center}

\begin{center}
if \( f - h < E \) then
\begin{center}
g := h; \quad h := h \times r; \quad \textbf{go to repeat; } \end{center}
\end{center}

\begin{center}
\textbf{end \textbf{if } Q[m, i] := g + h - f; } \textbf{end begin } \end{center}

end QPSH

\section*{CERTIFICATION OF ALGORITHM 181 [S15]}

\subsection*{COMPLEMENTARY ERROR FUNCTION—LARGE X}

[Henry C. Thacher, Jr., Comm. ACM 6 (June 1963), 315]

I. Clausen and L. Hansson (Reed. 20 Aug. 1964) DAEc, Risø, Denmark.

The procedure erfcz was tested in Gren-Algol with 29 significant bits and the number-range \( abs(x) < 2 \times 512 \) (approx. \( 1.3 \times 154 \)). The statement \( m := R := 0 \); was corrected to \( m := 0; \quad R := 0 \); [Because \( m \) and \( R \) are of different type; cf. Sec. 4.2.4 of the Algol Report, Comm. ACM 6 (Jan. 1963), 1-17.—Ed.] After this the tests were successful. The procedure was checked a.o. for \( x = 1.19 \times (-0.01) \) 0.72. The differences from table values increased from \( \text{io-8 at } x = 1.1 \) to \( \text{io-8 at } x = 0.75 \). Overflow occurred at \( x = 0.71 \).

\section*{CERTIFICATION OF ALGORITHM 224 [F3]}

\subsection*{EVALUATION OF DETERMINANT}

Leo J. Rotenberg, Comm. ACM 7 (Apr. 1964), 243


The "Evaluation of Determinant" program was tested on an Algol 60 compiler for an IBM 709 (Share distribution s3032). When the 10th line on page 254 was changed to read:

\begin{center}
\textbf{begin if } imax = r \textbf{ then go to resume else } \end{center}
correct results were obtained. It was tested up through 4 \times 4

\section*{CERTIFICATION OF ALGORITHM 237 [AI]}

\subsection*{GREATEST COMMON DIVISOR [J. E. Peck, Comm. ACM 7 (Aug. 1964), 481]}

T. A. Bray (Reed. 8 Sept. 1964) Boeing Scientific Research Laboratories, Seattle, Washington

This procedure was translated into the Fortran IV language and tested on the Univac 1107. No corrections were required and the procedure gave correct results for all cases tested.

\section*{Revised Algorithms Policy • May, 1964}

A contribution to the Algorithms department must be in the form of an algorithm, a certification, or a remark. Contributions should be sent in duplicate to the editor, typewritten double-spaced in capital and lower-case letters. Authors should carefully follow the style of this department, with especial attention to indentation and completeness of references. Material to appear in boldface type should be underlined in black. Blue underlining may be used to indicate italic type, but this is usually best left to the Editor.

An algorithm must be written in the Algol 60 Reference Language [Comm. ACM 6 (Jan. 1963), 1-17], and normally consists of a commented procedure declaration. Each algorithm must be accompanied by a complete driver program in Algol 60 which generates test data, calls the procedure, and outputs test answers. Moreover, selected previously obtained test answers should be given in comments in either the driver program or the algorithm. The driver program may be published with the algorithm if it would be of major assistance to a user.

Input and output should be achieved by procedure statements, using one of the following five procedures (whose body is not specified in Algol):

\begin{center}
procedure \texttt{inreal (channel, destination)}: \texttt{value channel; integer channel; \texttt{real destination}; comment the number read from channel is assigned to the variable destination; \ldots; } \end{center}

\begin{center}
procedure \texttt{outreal (channel, source)}: \texttt{value channel, source; integer channel, source; \texttt{real source}; comment the value of expression source is output to channel \texttt{channel}; \ldots; } \end{center}

\begin{center}
procedure \texttt{ininteger (channel, destination)}: \texttt{value channel; integer channel, destination; \ldots; } \end{center}

\begin{center}
procedure \texttt{outinteger (channel, source)}: \texttt{value channel, source; integer channel, source; \ldots; } \end{center}

\begin{center}
procedure \texttt{instring (channel, source)}, \texttt{value channel, source; integer channel, source; string channel; \ldots; } \end{center}

If only one channel is used by the program, it should be designated by \( 1 \) Examples:

\begin{center}
\texttt{outstring (1, \textit{\texttt{t'}, \texttt{w')); \	exttt{outreal (1, \texttt{t}); \texttt{for i := 1 step 1 until } \texttt{d =} \texttt{10 001 (1, A3)); \texttt{ininteger (1, digit [17]); } \end{center}

It is intended that each published algorithm be a well-organized, clearly commented, syntactically correct, and a substantial contribution to the Algol literature. All contributions will be refereed both by human beings and by an Algol compiler. Authors should give great attention to the correctness of their programs, since referees cannot be expected to debug them. Because Algol compilers are often incomplete, authors are encouraged to indicate in comments whether their algorithms are written in a recognized subset of Algol 60 ([see "Report on Subset Algol 60 (IFIP)," Comm. ACM 5 (Oct. 1964), 626-627].)

Certifications and remarks should add new information to that already published. Readers are especially encouraged to test and certify previously uncertified algorithms. Rewritten versions of previously published algorithms will be refereed as new contributions, and should not be imbedded in certifications or remarks.

Galley proofs will be sent to the authors; obviously rapid and careful proofreading is of paramount importance.

Although each algorithm has been tested by its author, no liability is assumed by the contributor, the editor, or the Association for Computing Machinery in connection therewith.

The reproduction of algorithms appearing in this department is explicitly permitted without any charge. When reproduction is for publication purposes, reference must be made to the algorithm author and to the Communications issue bearing the algorithm. —G.E.F.