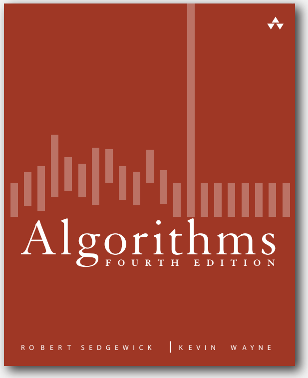
3.2 BINARY SEARCH TREES



- BSTs
- ordered operations
- deletion

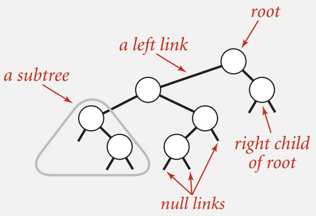
- **BSTs**
- ordered operations
- deletion

Binary search trees

Definition. A BST is a binary tree in symmetric order.

A binary tree is either:

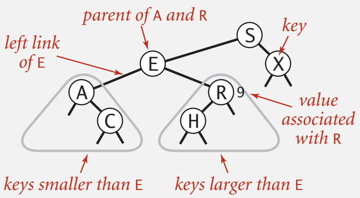
- Empty.
- Two disjoint binary trees (left and right).



Anatomy of a binary tree

Symmetric order. Each node has a key, and every node's key is:

- Larger than all keys in its left subtree.
- Smaller than all keys in its right subtree.



Anatomy of a binary search tree

BST representation in Java

Java definition. A BST is a reference to a root `Node`.

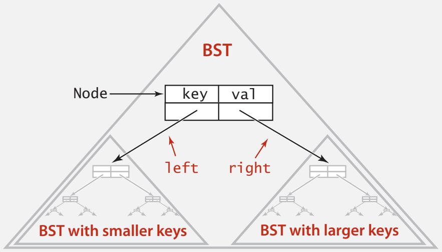
A `Node` is comprised of four fields:

- A `Key` and a `Value`.
- A reference to the left and right subtree.

smaller keys larger keys

```
private class Node
{
   private Key key;
   private Value val;
   private Node left, right;
   public Node(Key key, Value val)
   {
      this.key = key;
      this.val = val;
   }
}
```

`Key` and `Value` are generic types; `Key` is `Comparable`



Binary search tree

BST implementation (skeleton)

```
public class BST<Key extends Comparable<Key>, Value>
{
    private Node root;

   private class Node
   {  /* see previous slide */  }

   public void put(Key key, Value val)
   {  /* see next slides */  }

   public Value get(Key key)
   {  /* see next slides */  }

   public void delete(Key key)
   {  /* see next slides */  }

   public Iterable<Key> iterator()
   {  /* see next slides */  }

}
```

root of BST

BST search and insert demo

BST search: Java implementation

Get. Return value corresponding to given key, or `null` if no such key.

```
public Value get(Key key)
{
   Node x = root;
   while (x != null)
   {
      int cmp = key.compareTo(x.key);
      if      (cmp  < 0) x = x.left;
      else if (cmp  > 0) x = x.right;
      else if (cmp == 0) return x.val;
   }
   return null;
}
```

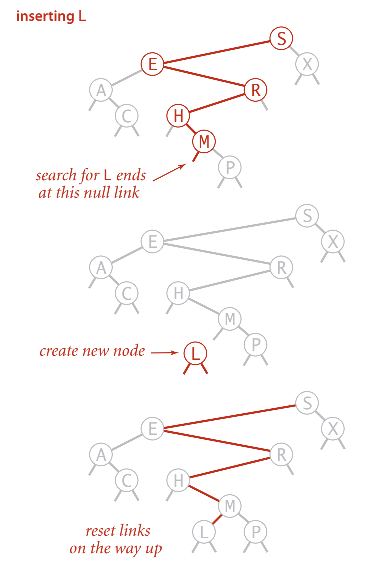
Cost. Number of compares is equal to 1 + depth of node.

BST insert

Put. Associate value with key.

Search for key, then two cases:

- Key in tree ⇒ reset value.
- Key not in tree ⇒ add new node.



search for L ends at this null link

create new node

reset links on the way up

Insertion into a BST

BST insert: Java implementation

Put. Associate value with key.

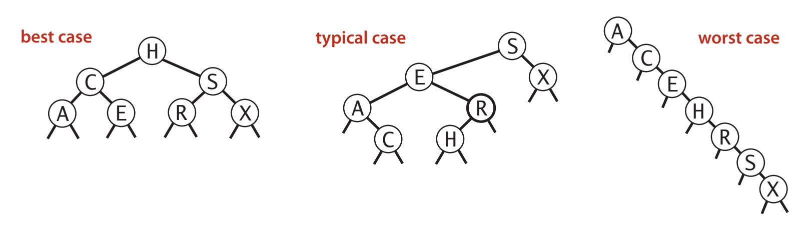
```
public void put(Key key, Value val)
{  root = put(root, key, val);  }

private Node put(Node x, Key key, Value val)
{
   if (x == null) return new Node(key, val);
   int cmp = key.compareTo(x.key);
   if      (cmp  < 0)
      x.left  = put(x.left,  key, val);
   else if (cmp  > 0)
      x.right = put(x.right, key, val);
   else if (cmp == 0)
      x.val = val;
   return x;
}
```

concise, but tricky, recursive code; read carefully!

Cost. Number of compares is equal to 1 + depth of node.

Tree shape

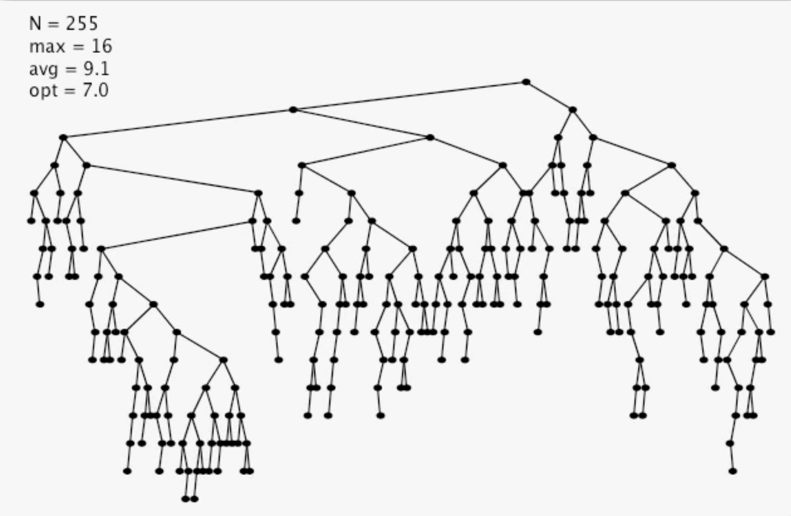
- Many BSTs correspond to same set of keys.
- Number of compares for search/insert is equal to 1 + depth of node.



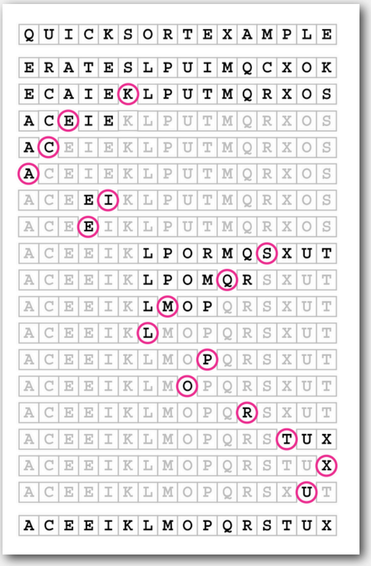
Remark. Tree shape depends on order of insertion.

BST insertion: random order visualization

Ex. Insert keys in random order.



Correspondence between BSTs and quicksort partitioning



Q	U	I	C	K	S	O	R	T	E	X	A	M	P	L	E
E	R	A	T	E	S	L	P	U	I	M	Q	C	X	O	K
E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
A	C	E	I	E	K	L	P	U	T	M	Q	R	X	O	S
A	C	E	I	E	K	L	P	U	T	M	Q	R	X	O	S
A	C	E	I	E	K	L	P	U	T	M	Q	R	X	O	S
A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
A	C	E	E	I	K	L	P	O	R	M	Q	S	X	U	T
A	C	E	E	I	K	L	P	O	M	Q	R	S	X	U	T
A	C	E	E	I	K	L	M	O	P	Q	R	S	X	U	T
A	C	E	E	I	K	L	M	O	P	Q	R	S	X	U	T
A	C	E	E	I	K	L	M	O	P	Q	R	S	X	U	T
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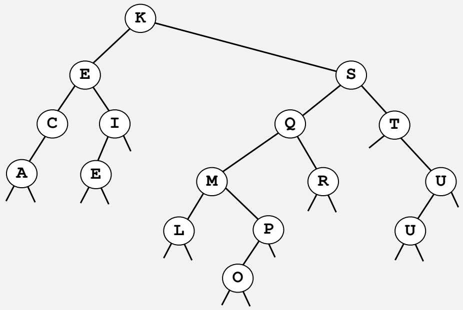
A	C	E	E	I	K	L	M	O	P	Q	R	S	X	U	T
A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
A	C	E	E	I	K	L	M	O	P	Q	R	S	X	U	T
A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X



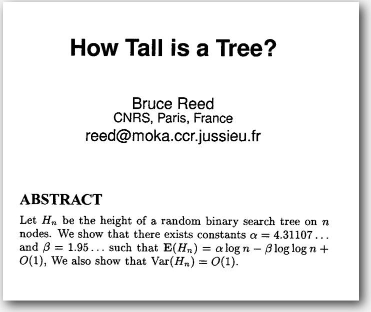
Remark. Correspondence is 1-1 if array has no duplicate keys.

BSTs: mathematical analysis

Proposition. If N distinct keys are inserted into a BST in random order, the expected number of compares for a search/insert is ~ $2 \ln N$.

Pf. 1-1 correspondence with quicksort partitioning.

Proposition. [Reed, 2003] If N distinct keys are inserted in random order, expected height of tree is ~ $4.311 \ln N$.



How Tall is a Tree?

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ABSTRACT

Let H_n be the height of a random binary search tree on n nodes. We show that there exists constants $\alpha = 4.31107\ldots$ and $\beta = 1.95\ldots$ such that $\mathrm{E}(H_n) = \alpha \log n - \beta \log \log n + O(1)$, We also show that $\mathrm{Var}(H_n) = O(1)$.

But… Worst-case height is N.
(exponentially small chance when keys are inserted in random order)

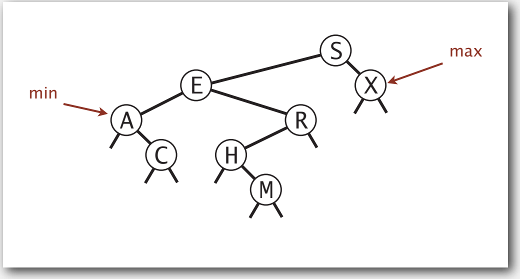
ST implementations: summary

implementation	guarantee		average case		ordered ops?	operations on keys
	search	insert	search hit	insert		
sequential search (unordered list)	N	N	N/2	N	no	`equals()`
binary search (ordered array)	lg N	N	lg N	N/2	yes	`compareTo()`
BST	N	N	1.39 lg N	1.39 lg N	?	`compareTo()`

- BSTs
- **ordered operations**
- deletion

Minimum and maximum

Minimum. Smallest key in table.
Maximum. Largest key in table.

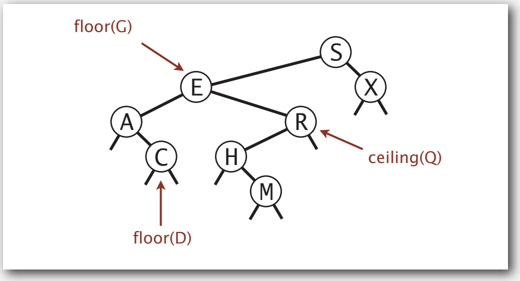


Q. How to find the min / max?

Floor and ceiling

Floor. Largest key ≤ to a given key.

Ceiling. Smallest key ≥ to a given key.



Q. How to find the floor /ceiling?

Computing the floor

Case 1. [k equals the key at root]

The floor of k is k.

Case 2. [k is less than the key at root]

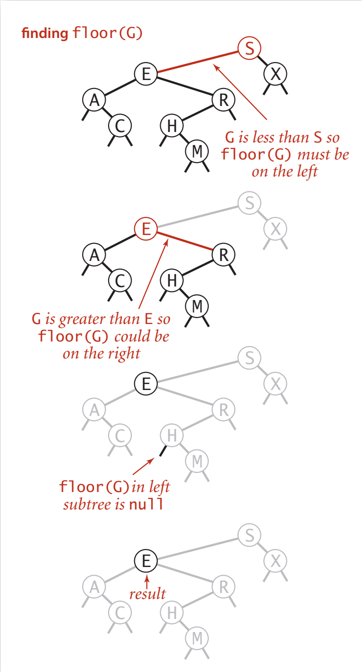
The floor of k is in the left subtree.

Case 3. [k is greater than the key at root]

The floor of k is in the right subtree

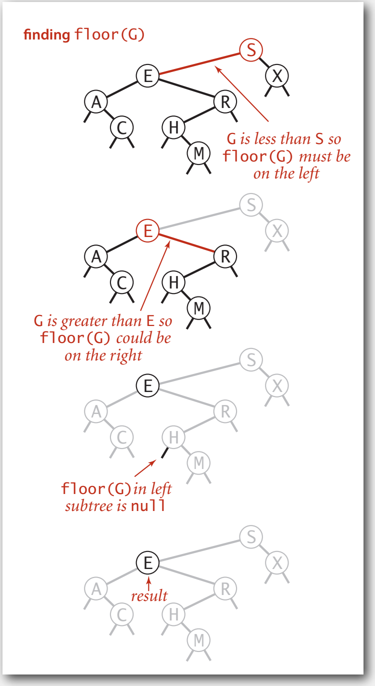
(if there is any key ≤ k in right subtree);

otherwise it is the key in the root.



Computing the floor

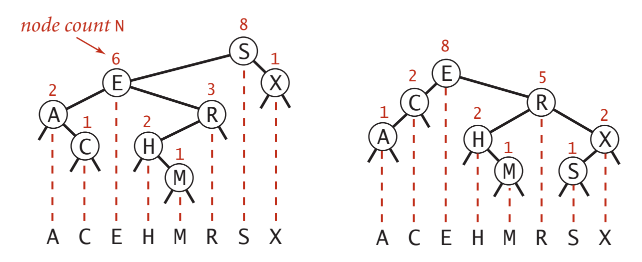
```
public Key floor(Key key)
{
   Node x = floor(root, key);
   if (x == null) return null;
   return x.key;
}
private Node floor(Node x, Key key)
{
   if (x == null) return null;
   int cmp = key.compareTo(x.key);

   if (cmp == 0) return x;

   if (cmp < 0)  return floor(x.left, key);

   Node t = floor(x.right, key);
   if (t != null) return t;
   else           return x;

}
```



Subtree counts

In each node, we store the number of nodes in the subtree rooted at that node.

To implement `size()`, return the count at the root.



Remark. This facilitates efficient implementation of `rank()` and `select()`.

BST implementation: subtree counts

```
private class Node
{
   private Key key;
   private Value val;
   private Node left;
   private Node right;
   private int N;
}
```

number of nodes in subtree

```
public int size()
{  return size(root);  }

private int size(Node x)
{
   if (x == null) return 0;
   return x.N;
}
```

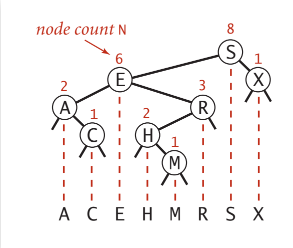
ok to call when x is null

```
private Node put(Node x, Key key, Value val)
{
   if (x == null) return new Node(key, val);
   int cmp = key.compareTo(x.key);
   if      (cmp  < 0) x.left  = put(x.left,  key, val);
   else if (cmp  > 0) x.right = put(x.right, key, val);
   else if (cmp == 0) x.val = val;
   x.N = 1 + size(x.left) + size(x.right);
   return x;
}
```

Rank

Rank. How many keys < k ?

Easy recursive algorithm (4 cases!)



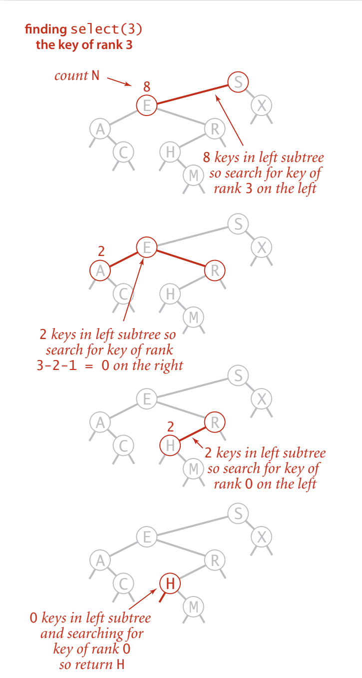
```
public int rank(Key key)
{  return rank(key, root);  }

private int rank(Key key, Node x)
{
   if (x == null) return 0;
   int cmp = key.compareTo(x.key);
   if      (cmp  < 0) return rank(key, x.left);
   else if (cmp  > 0) return 1 + size(x.left) + rank(key, x.right);
   else if (cmp == 0) return size(x.left);
}
```

Selection

Select. Key of given rank.

```
public Key select(int k)
{
    if (k < 0) return null;
    if (k >= size()) return null;
    Node x = select(root, k);
    return x.key;
}

private Node select(Node x, int k)
{
   if (x == null) return null;
   int t = size(x.left);
   if      (t  > k)
      return select(x.left,  k);
   else if (t  < k)
      return select(x.right, k-t-1);
   else if (t == k)
      return x;
}
```

finding select(3)
the key of rank 3



8 keys in left subtree so search for key of rank 3 on the left

2 keys in left subtree so search for key of rank 3-2-1 = 0 on the right

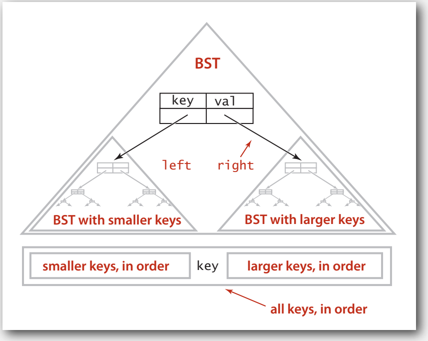
2 keys in left subtree so search for key of rank 0 on the left

0 keys in left subtree and searching for key of rank 0 so return H

Inorder traversal

- Traverse left subtree.
- Enqueue key.
- Traverse right subtree.

```
public Iterable<Key> keys()
{
    Queue<Key> q = new Queue<Key>();
    inorder(root, q);
    return q;
}

private void inorder(Node x, Queue<Key> q)
{
   if (x == null) return;
   inorder(x.left, q);
   q.enqueue(x.key);
   inorder(x.right, q);
}
```

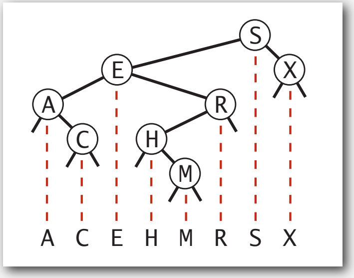


Property. Inorder traversal of a BST yields keys in ascending order.

Inorder traversal

- Traverse left subtree.
- Enqueue key.
- Traverse right subtree.

`inorder(S)`		S
`inorder(E)`		S E
`inorder(A)`		S E A
`enqueue A`	A	
`inorder(C)`		S E A C
`enqueue C`	C	
`enqueue E`	E	
`inorder(R)`		S E R
`inorder(H)`		S E R H
`enqueue H`	H	
`inorder(M)`		S E R H M
`enqueue M`	M	
`enqueue R`	R	
`enqueue S`	S	
`inorder(X)`		S X
`enqueue X`	X	
recursive calls	queue	function call stack



BST: ordered symbol table operations summary

	sequential search	binary search	BST
search	N	lg N	h
insert	1	N	h
min / max	N	1	h
floor / ceiling	N	lg N	h
rank	N	lg N	h
select	N	1	h
ordered iteration	N log N	N	N

h = height of BST
(proportional to log N
if keys inserted in random order)

order of growth of running time of ordered symbol table operations

- BSTs
- ordered operations
- **deletion**

ST implementations: summary

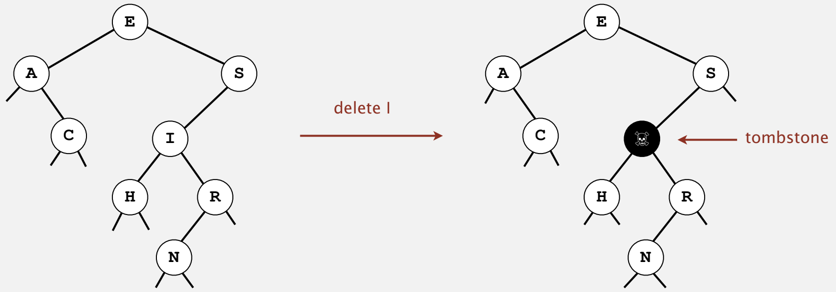
implementation	guarantee			average case			ordered iteration?	operations on keys
	search	insert	delete	search hit	insert	delete		
sequential search (linked list)	N	N	N	N/2	N	N/2	no	`equals()`
binary search (ordered array)	lg N	N	N	lg N	N/2	N/2	yes	`compareTo()`
BST	N	N	N	1.39 lg N	1.39 lg N	? ? ?	yes	`compareTo()`

Next. Deletion in BSTs.

BST deletion: lazy approach

To remove a node with a given key:

- Set its value to `null`.
- Leave key in tree to guide searches (but don't consider it equal to search key).



Cost. $\sim 2 \ln N'$ per insert, search, and delete (if keys in random order), where N' is the number of key-value pairs ever inserted in the BST.

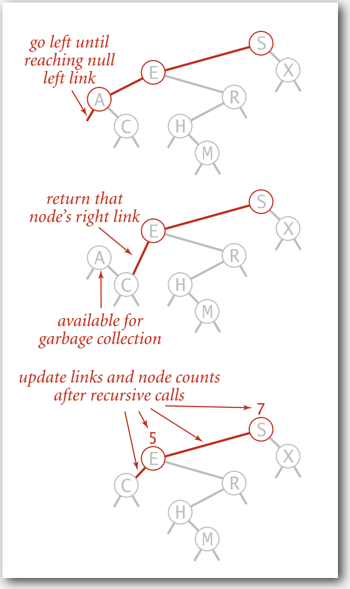
Unsatisfactory solution. Tombstone overload.

Deleting the minimum

To delete the minimum key:

- Go left until finding a node with a null left link.
- Replace that node by its right link.
- Update subtree counts.

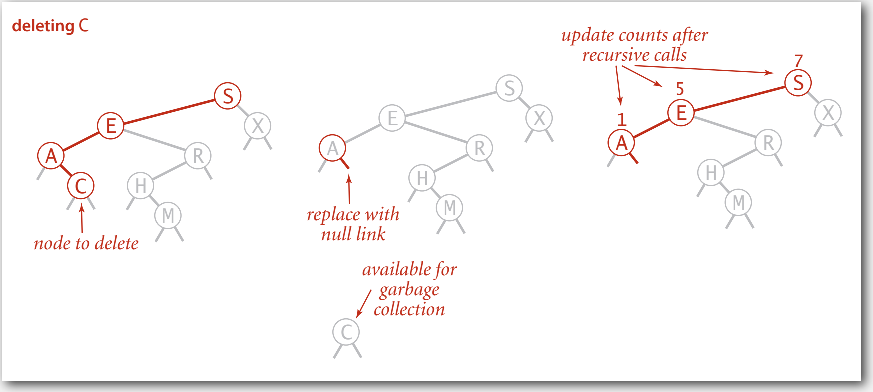
```
public void deleteMin()
{  root = deleteMin(root);  }

private Node deleteMin(Node x)
{
   if (x.left == null) return x.right;
   x.left = deleteMin(x.left);
   x.N = 1 + size(x.left) + size(x.right);
   return x;
}
```



Hibbard deletion

To delete a node with key k: search for node t containing key k.

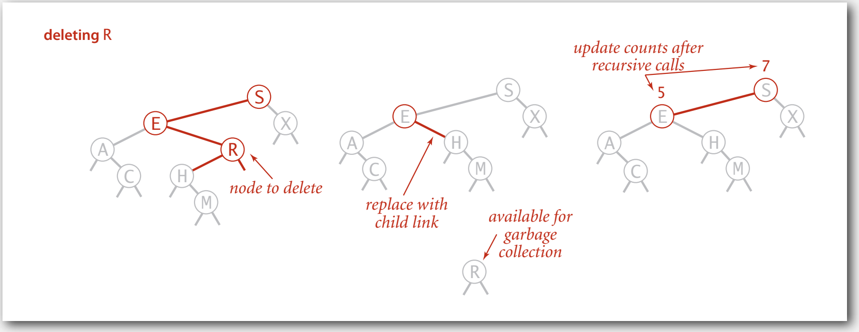
Case 0. [0 children] Delete t by setting parent link to null.



Hibbard deletion

To delete a node with key k: search for node t containing key k.

Case 1. [1 child] Delete t by replacing parent link.

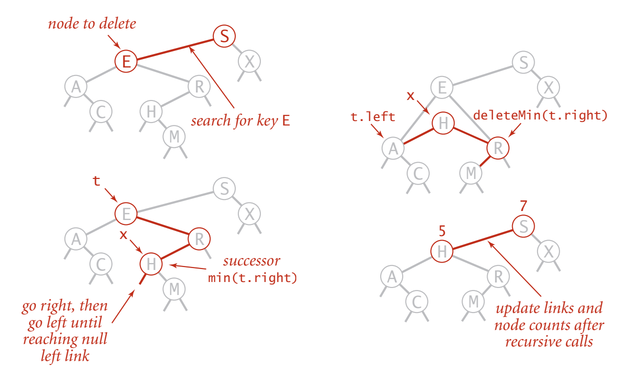


Hibbard deletion

To delete a node with key k: search for node t containing key k.

Case 2. [2 children]

- Find successor x of t. ← x has no left child
- Delete the minimum in t's right subtree. ← but don't garbage collect x
- Put x in t's spot. ← still a BST



Hibbard deletion: Java implementation

```
public void delete(Key key)
{  root = delete(root, key);  }

private Node delete(Node x, Key key) {
   if (x == null) return null;
   int cmp = key.compareTo(x.key);
   if      (cmp < 0) x.left  = delete(x.left,  key);
   else if (cmp > 0) x.right = delete(x.right, key);
   else {
      if (x.right == null) return x.left;

      Node t = x;
      x = min(t.right);
      x.right = deleteMin(t.right);
      x.left = t.left;
   }
   x.N = size(x.left) + size(x.right) + 1;
   return x;
}
```

search for key

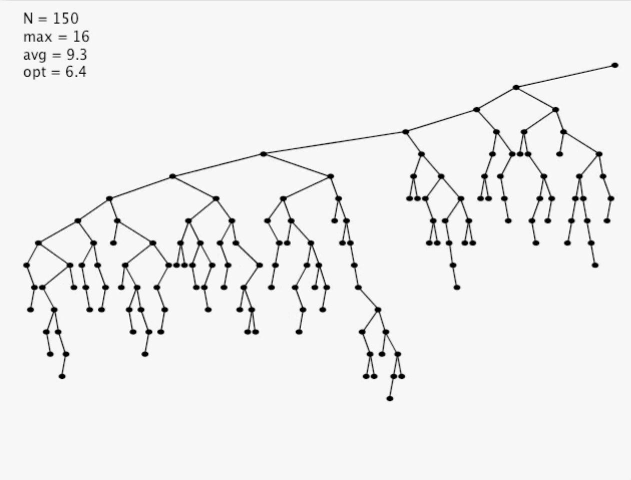
no right child

replace with successor

update subtree counts

Hibbard deletion: analysis

Unsatisfactory solution. Not symmetric.



Surprising consequence. Trees not random (!) ⇒ sqrt (N) per op.

Longstanding open problem. Simple and efficient delete for BSTs.

ST implementations: summary

implementation	guarantee			average case			ordered iteration?	operations on keys
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sequential search (linked list)	N	N	N	N/2	N	N/2	no	`equals()`
binary search (ordered array)	lg N	N	N	lg N	N/2	N/2	yes	`compareTo()`
BST	N	N	N	1.39 lg N	1.39 lg N	√N	yes	`compareTo()`

other operations also become √N if deletions allowed

Red-black BST. Guarantee logarithmic performance for all operations.